CONSISTENCY OF HESITANT FUZZY PREFERENCE RELATIONS

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In decision making it is necessary to study the consistency of preference relations to make sure that experts do not provide their preferences randomly. The lack of consistency might drive to wrong decisions. Taking into account the importance of checking the consistency in preference relations and the recent use of hesitant fuzzy sets to model the expert's hesitancy in decision making problems, the aim of this contribution is to propose and study definitions of consistency for hesitant fuzzy preference relations.

Keywords: consistency, preference relation, hesitant fuzzy set, decision making.

1. Introduction

Decision making is a daily activity for human beings in which they select the best alternative(s) among a set of alternatives [1]. To do so, experts involved in the problem express their preferences and knowledge to be processed in the decision process. Usually, the structure used to model the preferences elicitation is the preference relation [2], in which a pairwise comparison between alternatives is provided for each pair of alternatives. This makes easier the preference elicitation, because the stress of preference is focused on two alternatives at every preference provided by experts.

In decision making it is very important to study the consistency to make sure that the expert does not provide his/her preferences randomly. The lack of consistency in decision making could lead to inconsistent conclusions. Therefore, it is necessary to study the consistency in preference relations.

Recently, a novel concept to model the uncertainty provoked by hesitation, Hesitant Fuzzy Sets (HFSs) [3], has been introduced in the literature. It is an extension of fuzzy sets [4] able to model experts' hesitation when they provide their knowledge and opinions. This concept has been broadly applied in decision making [5, 6].

Taking into account the importance of providing consistent information and managing the experts' hesitation in decision making problems, the aim of this contribution is to study the consistency for Hesitant Fuzzy Preference Relations (HFPRs) introducing new definitions for weak and ordinal consistency and revising the existing ones for additive and multiplicative consistency [7, 8].

The structure of the paper is as follows. Section 2 revises some concepts about preference relations and hesitant information. Section 3 introduces the meaning of consistency and revises the classical consistency approaches. Section 4 revises and proposes new definitions of consistency for HFPRs. And finally, Section 5 points out some conclusions.

2. Preliminaries

This section revises in short some concepts about Fuzzy Preference Relations (FPRs) and hesitant information.

Definition 2.1. [9] A FPR P on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function,

$$\mu_P : X \times X \to [0, 1].$$

If the cardinality of X is small, the preference relation could be represented by the matrix $P = (p_{ij})_{n \times n}$ being $p_{ij} = \mu_P(x_i, x_j)$, $\forall i, j = \{1, \ldots, n\}$. p_{ij} represents the preference degree of the alternative x_i over alternative x_j. The preference matrix P is usually assumed to be reciprocal, i.e.,

$$p_{ij} + p_{ji} = 1, \forall i, j = \{1, \ldots, n\}. \tag{1}$$

In particular, $p_{ij} = 0.5$ indicates indifference between x_i and x_j; $0.5 < p_{ij} \leq 1$ indicates that x_i is strict preferred to x_j. the larger the value of p_{ij}, the greater the preference degree of the alternative x_i over x_j. If $p_{ij} = 1$ indicates that x_i is definitely preferred to x_j; $0 \leq p_{ij} < 0.5$ indicates that x_j is definitely preferred to x_i.

The use of FPRs has provided good results, however sometimes experts do no have enough information and knowledge about the decision problem and they might hesitate among several values when provide their preferences. In these situations, the use of the concept HFS is appropriate to model this type of uncertainty. A HFS is defined in terms of a function that returns a non-empty set of membership values represented as several possible values between 0 and 1 for each element in the domain.

Definition 2.2. [3] Let X be a reference set, a HFS on X is a function h that returns a subset of values in [0,1], $h : X \to \wp([0,1])$.

Note that a Hesitant Fuzzy Element (HFE) is a set of values in [0,1], and a HFS is a set of HFEs, one for each element in the reference set. More operations, extensions and applications of HFS can be found in [6].

The concept of HFS has been very useful in decision making problems, when experts involved in the problem have to provide their preferences and the use of only one value is not sufficient to represent that they want, because they doubt among multiple values. Thus, the concept of HFPR was presented in [10].

Definition 2.3. [10] Let $X = \{x_1, \ldots, x_n\}$ be a fixed set, then a HFPR H on X is represented by a matrix $H = (h_{ij})_{n\times n} \subset X \times X$, where $h_{ij} = \{h_{ij}^{\beta}, \beta = 1, 2, \ldots, \#h_{ij}\}$ ($\#h_{ij}$ is the number of values in h_{ij}) is a HFE that indicates all the possible preference degree(s) of the alternative x_i over x_j. In addition, h_{ij} should verify the following requirements: $h_{ij}^{\sigma(\beta)} + h_{ji}^{\sigma(\beta)} = 1$, $h_{ii} = \{0.5\}$, $\#h_{ij} = \#h_{ji}$, $i, j = \{1, \ldots, n\}$ and $h_{ij}^{\sigma(\beta)} < h_{ij}^{\sigma(\beta+1)}$, $h_{ji}^{\sigma(\beta+1)} < h_{ji}^{\sigma(\beta)}, i < j$ being $h_{ij}^{\sigma(\beta)}$ the β^{th} smallest value in h_{ij}.

3. Consistency of fuzzy preference relations

This section explains the meaning of consistency in a preference relation and revises different types of consistency for FPR.

Consistency of preference relations is associated with the transitivity property. The lack of consistency in decision making usually drives to inconsistent conclusions. There are two common types of consistency for a preference relation, ordinal consistency and cardinal consistency.

- *Ordinal consistency*: It is defined in terms of acyclicity, that means, if A is preferred to B, and B is preferred to C, then A should be preferred to C. This is called *weak consistency*.
- *Cardinal consistency*: This concept is more strict than ordinal consistency and it indicates the intensity of the preference value in pairwise comparisons, for instance, if A is two times better than B, and B is four times better than C, then A is eight times better than C. *Additive* and *multiplicative* consistency are two types of cardinal consistency.

Note that if a preference relation is perfectly cardinal consistent, it is also ordinal consistent but not vice versa. However, ordinal inconsistency always implies cardinal inconsistency, but not vice versa.

Once the meaning of consistency has been introduced, the different types of consistency, such as weak, ordinal, additive and multiplicative consistency for FPRs are revised.

Definition 3.1. [11] Let $P = (p_{ij})_{n\times n}$ be a FPR, $\forall i, j, k = \{1, 2, \ldots, n\}$, $i \neq j \neq k$, if $p_{ik} \geq 0.5$ and $p_{kj} \geq 0.5$, with $p_{ij} \geq 0.5$, then P satisfies the weak consistency.

It was pointed out that if the values of a FPR, P, verify the Def. 3.1, there could be contradictory assessments. For instance, let's suppose $p_{12} = 0.7$, $p_{23} = 0.5$ and $p_{13} = 0.5$, then p_{12}, p_{23}, p_{13} verify the weak consistency. Nevertheless, $p_{12} = 0.7$ that means $x_1 \succ x_2$, $p_{23} = 0.5$ indicates $x_2 \sim x_3$, therefore, it would be $x_1 \succ x_3$, but $p_{13} = 0.5$ which indicates $x_1 \sim x_3$ that contradicts $x_1 \succ x_3$. Consequently, Def. 3.1 does not work in some cases. The ordinal consistency of a FPR was introduced to overcome this problem.

Definition 3.2. [12] Let $P = (p_{ij})_{n\times n}$ be a reciprocal FPR, $\forall i, j, k = \{1, 2, \ldots, n\}$, $i \neq j \neq k$,

(1) if $p_{ik} > 0.5$ and $p_{kj} \geq 0.5$ or $p_{ik} \geq 0.5$, $p_{kj} > 0.5$ with $p_{ij} > 0.5$,
(2) if $p_{ik} = 0.5$ and $p_{kj} = 0.5$, with $p_{ij} = 0.5$.

then P verifies the ordinal consistency.

Another important consistency approaches are the additive and multiplicative consistency.

Definition 3.3. [13] Let $P = (p_{ij})_{n\times n}$ be a FPR, it is additive consistent if it verifies, $p_{ij} + p_{jk} = p_{ik} + 0.5$, $\forall i, j, k = \{1, 2, \ldots, n\}$.

Definition 3.4. [11] Let $P = (p_{ij})_{n\times n}$ be a FPR, it is multiplicative consistent if it verifies, $p_{ij}p_{jk}p_{ki} = p_{ji}p_{ik}p_{kj}$, $\forall i, j, k = \{1, 2, \ldots, n\}$.

4. Consistency of hesitant fuzzy preference relations

The definitions introduced in Section 3 can be extended in a simple way to deal with HFPR. This section proposes new definitions for weak and ordinal consistency, because the existing ones do not work in some cases and revises the definitions for additive [8] and multiplicative consistency [7].

In [14] was introduced the definition of weak consistency for HFPR, but it has several limitations, because it cannot be used when a HFE is equal to 0.5 and it needs that there is not circular triad in a hesitant fuzzy preference graph. To overcome such constraints new definitions for weak and ordinal consistency are proposed.

Definition 4.1. Let $H = (h_{ij})_{n\times n}$ be a HFPR, $\forall i,j,k = \{1,2,\ldots,n\}$, $i \neq j \neq k$, if there is $h_{ik}^{\sigma(\beta)} \geq 0.5$ and $h_{kj}^{\sigma(\beta)} \geq 0.5$, then $h_{ij}^{\sigma(\beta)} \geq 0.5$ ($h_{ij}^{\sigma(\beta)} \in h_{ij}$), therefore H satisfies the weak consistency.

Definition 4.2. Let $H = (h_{ij})_{n\times n}$ be a HFPR, $\forall i,j,k = \{1,2,\ldots,n\}$, $i \neq j \neq k$,

(1) if there is $h_{ik}^{\sigma(\beta)} > 0.5$, $h_{kj}^{\sigma(\beta)} \geq 0.5$, or $h_{ik}^{\sigma(\beta)} \geq 0.5$, $h_{kj}^{\sigma(\beta)} > 0.5$, with $h_{ij}^{\sigma(\beta)} > 0.5$,

(2) if there is $h_{ik}^{\sigma(\beta)} = 0.5$ and $h_{kj}^{\sigma(\beta)} = 0.5$, with $h_{ij}^{\sigma(\beta)} = 0.5$.

then H is ordinal consistent.

Before introducing the definitions for additive and multiplicative consistency, it is necessary to highlight that in a HFPR the number of pairwise comparison could be different, because of a HFE could have several membership degrees. To cope this problem, there are two different approaches: (i) it adds values in the HFPR and obtains a Normalized Hesitant Fuzzy Preference Relation (NHFPR), in which all the HFEs are formed by the same number of membership degrees [7], (ii) it removes elements of the HFPR by means of an error analysis to obtain a FPR with the highest consistency level [15]. Taking into account that the goal of this contribution is to study the ordinal and cardinal consistency of HFPR, we focus on the former approach which is used in the next definitions.

Definition 4.3. [8] Let $H = (h_{ij})_{n\times n}$ be a HFPR and $\bar{H} = (\bar{h}_{ij})_{n\times n}$ its NHFPR with the optimized parameter ς $(0 \leq \varsigma \leq 1)$, $\forall i,j,k = \{1,2,\ldots,n\}$, $i \neq j \neq k$, if $\bar{h}_{ij}^{\rho(\beta)} = \bar{h}_{ik}^{\rho(\beta)} - \bar{h}_{jk}^{\rho(\beta)} + 0.5$ being $\bar{h}_{ij}^{\rho(\beta)}$ the β^{th} smallest element in $\bar{h}_{ij}$, then H satisfies the additive consistency with ς.

Definition 4.4. [7] Let $H = (h_{ij})_{n\times n}$ be a HFPR and $\bar{H} = (\bar{h}_{ij})_{n\times n}$ its NHFPR with the optimized parameter ς $(0 \leq \varsigma \leq 1)$, $\forall i,j,k = \{1,2,\ldots,n\}$, $i \neq j \neq k$, if $\bar{h}_{ik}^{\rho(\beta)}\bar{h}_{kj}^{\rho(\beta)}\bar{h}_{ji}^{\rho(\beta)} = \bar{h}_{ki}^{\rho(\beta)}\bar{h}_{ij}^{\rho(\beta)}\bar{h}_{jk}^{\rho(\beta)}$ being $\bar{h}_{ij}^{\rho(\beta)}$ the β^{th} smallest element in $\bar{h}_{ij}$, then H satisfies the multiplicative consistency with ς.

5. Conclusions

It is important to consider the consistency in decision making to overcame inconsistent conclusions. The use of hesitant fuzzy sets is a suitable tool to model the experts' hesitation and it has been widely used in decision making. This contribution proposes and revises different types of consistency for hesitant fuzzy preference relations.

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